I. Applications of Integrals

A. Area of a region

If a region lies between two curves the area of the region would be the area area of the region under the graph of g subtracted from the region under the graph of f.Therefore if f and g are continuous on [a,b] and g(x) $\leq$f(x) for all x in ([a,b],then the area of the region bounded by the graphs of f and g and the vertical lines:x=a,X=b is

A= $\int_a^b f(x)-g(x)dx$

where you always subtract the bottom function from the top function.
if the curves are with respect to the y-axis ,subtract right most curve minus the left curve.

Area of a region Between two intersecting Curves

You simply use the same equation as the area under a curve but if two curves,but if two curves intersect at more than two points,you must find all points of intersection and check to see which curve is above the outer in each interval determined by this points

Ex:Find the area bounded by the graph of $y=x^2+2$,y=-x, x=0, x=1.

1.Start by graphing

A= $\int_a^b [f(x)-g(x)]dx$

$=\int_0^1 [(x^2+2)-(-x)]dx$

$=\int_0^1 [x^2+x+2]dx$

$=\frac{x^3}{3}+\frac{x^2}{2}+2x\,]_0^1$

$=\frac{1}{3}+\frac{1}{2}+2$

$=\frac{17}{6}$

The volume of a Known cross sections

There are three methods to do this:1**.Disk Method**
2.Washer method
3. Shell Method

1.**The disk method**

mostly used to find the volume of a solid formed by revolving a plane region about about the indicated axis.The best application of disk method is a graph that involves a plane region

bounded by the graph of f ant the x-axis/y-axis.If the axis of revolution is the x-axis,the radius R(x) is simply f(x) the disk formula is

$\pi \int_a^{\pi} [R(x)]^2 dx$ or $\pi \int_a^{\pi} [R(y)]^2 dy$ where R is the radius of the cylinder

horizontal axis of revolution vertical axis of revolution

r(x)is the distance from the axis to the function.the rectangle must be attached to the axis of rotation.the axis of rotation determines whether the integral is in terms of x or y.

$\pi r^2 =$ area of cross-section.

the rectangle must be attached to the axis of rotation

Ex:find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \sqrt{sin(x)}$,and the x-axis $(0 \leq x \leq \pi$)

1.graph the equation

from the representative rectangle you can see that the r(x)=$f(x) = \sqrt{sin(x)}$

therefore the volume is

V= $\pi \int_0^{\pi} (\sqrt{sin(x)^2}$

= $\pi \int_0^{\pi} sin(x) dx$

=$\pi[-cosx]\pi$-0

=$\pi(1+$1)

=2π

<u>The Washer Method:</u>

this method is mostly used to find the volume of a region with holes.a washer is formed by revolving a rectangle about an axis ,as shown in fig c.the volume of the solid of revolution,consider a region bounded by an outer radius R(x) and the inner radius r(x).The volume of the resulting solid will be represented by

V=$\pi \int^{\pi} ([R(x)]^2 - [r(x)]^2) dx$

Ex:find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$,$y = x^2$ about the x-axis

R(x)=$\sqrt{x}$

r(x)=x^2

V= $\pi \int_a^b ([R(x)]^2 - [r(x)]^2)dx$

= $\pi \int^1 [(\sqrt{x})^2 - (x^2)^2]dx$

= $\pi \int_0^1 (x - x^4)dx$

= $\pi[\frac{x^2}{2} - \frac{x^5}{5}]$0-1

= $\frac{3\pi}{10}$

For cross sections of area A(x) taken perpendicular to the x-axis

V=

For cross sections of area A(y) taken perpendicular to the y-axis

V=

The Shell Method:

consider a representative rectangle,with w as the width of the triangle and h is the height of the triangle and p is the distance between the axis of revolution and the center if the rectangle.when this rectangle is revolved it forms a cylindrical shell. to find the volume of this shell,consider two cylinders.the radius of the larger cylinder corresponds with the outer radius of the shell,and the radius of the smaller cylinder corresponds to the radius of the inner shell.because p is the average radius of the shell,you know the outer radius is p+(w/2) and the inner radius is p-(w/2)

the formula to shell method is

V= $2\pi \int_a^b r(x)h(x)dx$

To know whether to use the disk or shell method the disk and shell methods.**Remember:**for the disk method,the representative rectangle is perpendicular to the axis of revolution,whereas the shell method, the representative rectangle is parallel to the axis of revolution. To find the volume of a solid of revolution with shell method.

The Average value of a function:

The average value theorem relates the average value of the function to the average all the instantaneous values

Let f be a function in **[a, b]**. The **average value** of **f** from **x = a** to **x = b** is the integral

$$\frac{1}{b-a} \int_a^b f(x)dx$$

Example: Find the average value of y = sin x between x = 0 and x = p is

function:

$$= \frac{1}{\pi - 0} \int_0^{\pi} sin(x)dx$$

$$= \frac{1}{\pi}(-cos(x)$$

$$= \frac{1}{\pi}(1+1)$$

$$= \frac{2}{\pi}$$

The Distance Travelled by a Particle Along a Line

(This section cannot be found in the book)

This is the study of motion -- position, velocity and acceleration -- along a graph.

Position is $x(t)$. It determines where the particle is located on the x-axis at any time t.

Velocity is $v(t) = x'(t)$. It determines how fast the position is changing at a time t as well as the direction of movement.

Acceleration is $a(t) = v'(t) = x''(t)$. It determines how fast the velocity is changing at time t; the sign indicates if the velocity is increasing or decreasing.

If the velocity of the particle is positive, it is moving to the right. If its velocity is negative, it is moving to the left. If the acceleration of the is positive, the velocity is increasing. If the acceleration is negative, the velocity is decreasing. In order for the particle to change direction, the velocity must change signs.

“Initially” means when $t = 0$

“At the origin” means when the position, $x(t) = 0$

“At rest” means when the velocity, $v(t) = 0$

To find the average velocity over a time interval, divide the change in position by the change in time.

Speed is the absolute value of velocity.

One way to determine total distance travelled over a time interval is to find the sum of the absolute values of the differences in position between all resting points. This can also be done by using integrals. There are three ways to use an integral in the study of motion:

1. $\int v(t)dt$ As this is an indefinite integral, it will produce an expression for position at time t. There will be a **+C**, whose value can be determined if you know a position value at a certain time.

2. $\int_{t_1}^{t_2} v(t)dt$ As this is a definite integral, the number produced will represent a change in position over the time interval (the displacement. Displacement is the difference between the initial position and the final position of an object). By the Fundamental Theorem of Calculus, since $v(t) = x'(t)$, the integral will yield $x(t_2) - x(t_2)$. It may be positive or negative depending on if the particle lands to the right or left of its original starting position.

3. $\int_{t_1}^{t_2} |v(t)|dt$ This definite integral will produce a number representing the total distance traveled by the particle over the time interval. The answer should always be positive.

For Example:

(i) The velocity function of a particle is $v(t) = 5t+2$. It is at a point P = 6 , 3 seconds after it leaves its initial position. What is the position function for this particle?

First, find $\int v(t)dt$.

$$\int (5t+2)dt = \frac{5t^2}{2} + 2t + C$$

To find the arbitrary constant, we must plug and chug the initial conditions:

$$6 = \frac{5(3)^2}{2} + 2(3) + C$$
$$C = -22.5$$

Therefore, the position function of this particle is equal to:

$$P(t) = \frac{5t^2}{2} + 2t - 22.5$$

(ii) The rate of change, in kilometers per hour, of the altitude of a hot air balloon is given by $r(t) = t^3 - 4t^2 + 6$ for time $0 \le t \le 4$, where t is measured in hours. Assume the balloon is initially at ground level.

Find the net change in the altitude of the balloon over the time interval $0 \le t \le 4$.

As we want to find the net change (ie. displacement) in altitude of the balloon, we use the

displacement integral:

$$\int_{t_1}^{t_2} v(t)dt \Rightarrow \int_0^4 (t^3 - 4t^2 + 6)dt = \frac{t^4}{4} - \frac{4t^3}{3} + 6t$$ to be evaluated from t=4 to t=0.

$$= \frac{(4)^4}{4} - \frac{4(4)^3}{3} + 6(4) = 2.667$$ km.

(ii) Find the total vertical distance traveled by the balloon over the time interval $0 \le t \le 4$.

As we want to find the total distance travelled, we use the total distance integral:

$$\int_{t_1}^{t_2} |v(t)|dt \Rightarrow \int_0^4 (|t^3 - 4t^2 + 6)dt| = \left|\frac{t^4}{4} - \frac{4t^3}{3} + 6t\right|$$ to be evaluated from t=4 to t=0.

$$= \left|\frac{(4)^4}{4} - \frac{4(4)^3}{3} + 6(4)\right|$$ = 11.529 km.

F. Finding an Accumulative Change from a Rate of Change(Chapter 4-7)

The basic idea behind finding the accumulative rate of change is if you take the Velocity equation for a particular question, which means that you know how fast something is traveling, if you integrate that equation, you will be able to find the distance it has traveled. For example, if you have a bucket that water is being poured into, and if you know how fast that water is being poured, you can take that equation and integrate it and you will find the distance the water has traveled from its original container to the bucket.

- An Antiderivative is the inverse of a derivative.
- A General Solution (also known as a is an indefinite integral) that gives a family of solutions to the integral, all with the same shape, just shifted vertically.
- A Particular Solution (also known as an definate integral) is an integral that only has one answer and you find it by using the initial condition given.

$$\frac{dy}{dx} = ky$$
$$\int \frac{dy}{y} = \int kdt$$
$$e^{In|y||} = e^{kt+C}$$
$$y = Ce^{kt}$$

II. Applications of Antidifferentiation

A. Finding Specific Antiederivatives Using Initial Conditions (Chapter 6 Section 2 & 3)

- A Particular Solution uses initial conditions to find C (the Arbitrary constant) with a specific Differential Equations.
- A Differential Equation is any equation that contains a derivative.
- The Arbitrary Constant-C, is used in all indefinite integrals.

In Order to find a Specific anti-derivative from an equation you must:

1. Move all common variables to the same side, this may include separating the variables.

Which means taking the $\frac{dy}{dx}$ and multiplying the both sides by dx.

$y' = 2x \qquad y(4) = 2$

$\frac{dy}{dx} = 2x$

$dy = 2xdx$

2. Then integrate each side separately making sure to add a '+ C'. Integrating the equation may include having to do a U-substitution.

$\int dy = \int 2xdx$

$y = x^2$ +C

U-Substation

$\int \frac{x}{x^2+1} dx \qquad U = x - 1 \qquad Du = dx$

$\frac{1}{2} \int \frac{2x}{x^2+1} = \frac{du}{u} \qquad \frac{1}{2} ln|u| + C$

$\frac{1}{2} ln(x^2+1) + C$

3. From there use the initial condition to find C.

$y(4) = 2$

$y = x^2 + C \qquad 2 = (4)^2 + C$

$-14 = C$

4. Plug C back into the original equation to get your result.

$y = x^2 - 14$

B. Solving Separable Differential Equations

(This section can be found on pages 421-431 of our book)

Solving separable differential equations is relatively simple, yet crucial to solving exponential problems.

A differential equation is an equation that contains a derivative.

To solve a separable differential equation and integrate to find a solution:

$y' = \frac{5x}{y}$

1. Put the equation into terms of implicit differentiation.

$$\frac{dy}{dx} = \frac{5x}{y}$$

2. Put the variables on one side.

$$y\,dy = 5x\,dx$$

3. Integrate each side separately. Put +C on just one side (usually the side with the independent variable).

$$\int y\,dy = \int 5x\,dx$$

$$\frac{y^2}{2} = \frac{5x^2}{2} + c$$

4. Decide if the question is asking for a general solution or a particular solution.
A) If asking for a general solution, just simplify your answer! Thus,

$$y^2 = 2\left(\frac{5x^2}{2}\right) + C$$

$$y^2 = \frac{10x^2}{2} + C$$

B) If asking for a particular solution, use the given information to solve.
If, for instance, we were told that this function passes through the points (2,4), then we would plug and chug to solve for C:

$$(4)^2 = \frac{10(2)^2}{2} + C$$

$$16 - 20 = C$$

$$C = -4$$

Thus:

$$y^2 = \frac{10x^2}{2} - 4$$

Examples:

(i) Find the general solution to $\frac{dr}{ds} = 0.05r$

First, we put the equation into terms of implicit differentiation (already done). Then w e put the variables on one side.

$$\frac{dr}{r} = 0.05\,ds$$

Then we integrate each side, and put +C on only one side.

$$ln(r) = 0.05s + C$$

We then simplify, and we're done!

$$r = Ce^{0.05s}$$

(ii) Find the particular solution of $yy' - e^x = 0$with the initial condition of $y(0) = 4$.

Again, we begin by putting the equation into terms of implicit differention and then put the variables on one side:

$$ydy = e^x dx$$

Then we integrate and leave +C on only one side and then simplify:

$$\frac{y^2}{2} = e^x + C \Rightarrow y^2 = 2(e^x + C)$$

We then use the initial condition $y(0) = 4$to solve for C:

$$(4)^2 = 2(e^{(0)} + C) \Rightarrow 16 - 2 = C = 14$$

Then we plug C back into the original equation:

$$y^2 = 2(e^x + 14)$$

2. Using Separable Differential Equations in Modelling
(This section can be found on pages 364-370 of our book)

Separable differential equations are used in various real-life applications. Bankers, scientists, park rangers and math teachers all use these equations in their lines of work. Whether to predict money in a bank account or the mass of a bacteria culture at a particular time, these formulas aid professionals in every area. We use separate differentials to figure out

exponential formulas.

(i) The equation $y' = kt$

Using the method of separating and solving differentiable equations, the equation $y' = kt$ is solved so that:

$$\frac{dy}{dt} = ky$$

$$\int \frac{dy}{y} = \int k\,dt$$

$$ln|y| = kt + C$$

$$y = Ce^{kt}$$

Where C = the initial amount/population
k = the rate of growth/decay
t = time
y = the amount/population at a time t

This is the exponential growth equation. It is used in a variety of ways.
For example:
In a year, the number of cases of a disease decreases by 20%. There are 10,000 cases today. How many years will it take to reduce the number of cases to 1000?

$$y = Ce^{kt}$$

The initial amount of cases of disease is 10,000. Thus, C = 10,000.

$$y = 10000e^{kt}$$

We want to solve for k. We know that when t=1, the number of cases will be 80% of the initial value, i.e. 8000.

$$8000 = 10000e^{k(1)}$$

$$\frac{8000}{10000} = e^{k}$$

$$ln\left|\frac{8000}{10000}\right| = k$$

thus,

$k = ln\, 0.8$

To solve for the time it takes for the cases to reduce to 1000, we plug in 1000 for y and solve for t:

$$1000 = 10000e^{(ln\, 0.8)t}$$

$$\frac{1000}{10000} = e^{(ln\, 0.8)t}$$

$$ln\,(\frac{1000}{10000}) = (ln\, 0.8)t$$

$$t = \frac{ln(\frac{1000}{10000})}{(ln\, 0.8)}$$

$t \approx 10.32$ years

(ii) Some other very important exponential formulas are:

(a) $y = \frac{L}{1 + be^{-kt}}$ Where

L= carrying capacity (the maximum population)
k = the rate at which the population is growing/decaying
t = time (acts like x, is the independent variable)
b = NOT the initial population! This is a different number that you will have to solve for in the beginning (usually by making t=0).
y = the population at a time t. May also be represented by P.

Use this equation when the population has a maximum (eg. the bacteria culture can weigh up to 20g, the game park can only host up to 200 lions, etc).

(b) $P = e^{rt}$ Where P= the money gained after time t.
r = the rate of interest
t = time (like x)

Use this equation when interest is compounded continuously.

Examples:
(i) A state game commission releases 40 elk into a game refuge. After 5 years, the elk

population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population p is:

$$\frac{dp}{dt} = kp(1 - \frac{p}{4000}), 40 \leq p \leq 4000$$

where t is the number of years.

(a) Write a model for the elk population in terms of t.
(b) Use the model to estimate the elk population after 15 years.

(a) We know that the maximum number of elk that the park can take is 4000. Thus, L= 4000. We can plug this into our equation:

$$y = \frac{4000}{1 + be^{-kt}}$$

Because we know that p(0) = 40, we can solve for b:

$$40 = \frac{4000}{1 + be^{-k(0)}} \Rightarrow 40 = \frac{4000}{1 + b} \Rightarrow b = 99$$

Because we know p = 104 when t =5, we can solve for k.

$$104 = \frac{4000}{1 + 99e^{-k(5)}} \Rightarrow k \sim 0.194$$

So, a model for the elk population can be given by:

$$p = \frac{4000}{1 + 99e^{-0.194t}}$$

(b) To estimate the elk population in 15 years, we can substitute 15 for t in the model:

$$p = \frac{4000}{1 + 99e^{-0.194(15)}} \Rightarrow p = 626$$

(ii) Find the time for an initial investment of $1000 to double in a savings account in which interest is compounded continuously. The interest rate is 6%.

We begin by using our equation for continuously compounded interest and plugging in what we can (we know that p= 2000 because 1000 doubled =2000).

$P = e^{rt} \Rightarrow 2000 = e^{6\%t}$

We then solve for t:

$\frac{ln(2000)}{6\%} = t = 1.268$ years.